JOURNAL OF THE
CHUNGCHEONG MATHEMATICAL SOCIETY
Volume **29**, No. 4, November 2016
http://dx.doi.org/10.14403/jcms.2016.29.4.553

PERIODICITIES OF SOME CHARACTERISTIC MATRICES OF CELLULAR AUTOMATA WITH RULE 60 AND INTERMEDIATE BOUNDARY CONDITION

Jae Gyeom Kim*

Abstract. We characterize periodicities of some characteristic matrices of cellular automata configured with rule 60 and intermediate boundary condition.

1. Introduction

Boundary conditions of cellular automata greatly influence properties of the cellular automata. Properties of cellular automata with intermediate boundary condition have been studied some researchers [1,2,4,5,7]. Recently, periodicities of some characteristic matrices of cellular automata configured with rule 60 and intermediate boundary condition was partially investigated [3].

In this note, we will characterize periodicities of some characteristic matrices of cellular automata configured with rule 60 and intermediate boundary condition.

2. Preliminaries

A cellular automaton (CA) is an array of sites (cells) where each site is in any one of the permissible states. At each discrete time step (clock cycle) the evolution of a site value depends on some rule (the combinational logic) which is a function of the present states of its k neighbors for a k-neighborhood CA. For a 2-state 3-neighborhood CA, the evolution of the (i)th cell can be represented as a function of the present states of $(i-1)$th, (i)th, and $(i+1)$th cells as: $x_i(t+1) =$

Received June 05, 2016; Accepted November 09, 2016.
2010 Mathematics Subject Classification: Primary 68Q80.
Key words and phrases: cellular automaton, intermediate boundary condition.

$f\{x_{i-1}(t), x_i(t), x_{i+1}(t)\}$, where f represents the combinational logic. For such a CA, the modulo-2 logic is always applied.

For a 2-state 3-neighborhood CA there are 2^3 distinct neighborhood configurations and 2^{2^3} distinct mappings from all these neighborhood configurations to the next states, each mapping representing a CA rule. The CA, characterized by a rule known as rule 60, specifies an evolution from the neighborhood configurations to the next states as;

$$\begin{array}{cccccccc} 111 & 110 & 101 & 100 & 011 & 010 & 001 & 000 \\ 0 & 0 & 1 & 1 & 1 & 1 & 0 & 0 \end{array} .$$

The rule name 60 comes from that 00111100 in binary system is 60 in decimal system. The corresponding combinational logic of rule 60 is

$$x_i(t+1) = x_{i-1}(t) \oplus x_i(t),$$

that is, the next state of (i)th cell depends on the present states of its left and self neighbors.

If in a CA the same rule applies to all cells, then the CA is called a uniform CA; otherwise the CA is called a hybrid CA. There can be various boundary conditions; namely, null (where extreme cells are connected to logic '0'), intermediate (where the 2nd right cell of the leftmost cell of a 3-neighborhood CA is assumed to be the left neighbor of the leftmost cell of the CA and the 2nd left cell of the rightmost cell of the CA is assumed to be the right neighbor of the rightmost cell of the CA), periodic (where extreme cells are adjacent), etc. The number of cells of a CA is called the length of a CA.

The characteristic matrix T of a CA is the transition matrix of the CA. The next state $f_{t+1}(x)$ of a linear CA is given by $f_{t+1}(x) = T \times f_t(x)$, where $f_t(x)$ is the current state and t is the time step. If all the states of the CA form a single or multiple cycles, then it is referred to as a group CA. One of basic characterizations of periodicities of characteristic matrices of cellular automata is Lemma 2.1.

Lemma 2.1 ([6]). *Let H be a uniform CA of length n configured with rule* 60 *and null boundary condition. If $2^{t-1} < n \leq 2^t$ for some positive integer t, then the group order of H is 2^t.*

From Lemma 2.1, Lemma 2.2 could be derived. And Theorem 2.3 was proved based on Lemma 2.2.

Lemma 2.2 ([3]). *Let T be the characteristic matrix of a uniform CA of length n with $3 + 2^{t-1} < n \leq 3 + 2^t$ for some positive integer t configured with rule* 60 *and intermediate boundary condition. If $T^{m+1} = T$ for some positive integer m, then m is a multiple of 2^t.*

Theorem 2.3 ([3]). *Let T be the characteristic matrix of a uniform CA of length n with $3 + 2^{t-1} < n \leq 3 + 2^t$ for some positive integer t configured with rule* 60 *and intermediate boundary condition. If $T^{m+1} = T$ for some positive integer m, then m is a multiple of $2^t \cdot 3$.*

3. Characteristic matrices of cellular automata

In this section, we deal with characteristic matrices of uniform CA of sufficiently large length configured with rule 60 and intermediate boundary condition. Such a matrix T of the CA is given by

$$T_{i,j} = \begin{cases} 1, & \text{if } i = j \text{ or } i = j+1, \\ 1, & \text{if } i = 1 \text{ and } j = 3, \\ 0, & \text{otherwise} \end{cases} \quad \text{or } T = \begin{pmatrix} 1 & 0 & 1 & & & & & \\ 1 & 1 & 0 & & & & & \\ & 1 & 1 & & & & & \\ & & 1 & 1 & & & & \\ & & & 1 & 1 & & & \\ & & & & \cdot & \cdot & & \\ & & & & & \cdot & \cdot & \\ & & & & & & \cdot & \cdot \end{pmatrix}$$

where all the values of the blank entries are zero. From now on, all the values of the blank entries in matrix representation will always be zero unless otherwise specified.

And we can have T^3 as follows;

$$T^3 = \begin{pmatrix} 0 & 1 & 1 & 0 & 0 & 0 & & & \\ 1 & 0 & 1 & 0 & 0 & 0 & & & \\ 1 & 1 & 0 & 0 & 0 & 0 & & & \\ 1 & 1 & 1 & 1 & 0 & 0 & & & \\ 0 & 1 & 1 & 1 & 1 & 0 & & & \\ 0 & 0 & 1 & 1 & 1 & 1 & & & \\ & & & \cdot & \cdot & \cdot & \cdot & & \\ & & & & \cdot & \cdot & \cdot & \cdot & \\ & & & & & \cdot & \cdot & \cdot & \cdot \end{pmatrix}.$$

Now we are interested in the matrices $(T^3)^{2^t}$ with non-negative integer t. In fact, $(T^3)^{2^0} = T^3$ and $(T^3)^{2^1}$ and $(T^3)^{2^2}$ are as follows;

$$(T^3)^{2^1} = T^6 = \left(\begin{array}{cccccccccccccc}
0 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & & & \\
1 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & & & \\
1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & & & \\
1 & 1 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & & & \\
1 & 1 & 1 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & & & \\
0 & 1 & 1 & 1 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & & & \\
1 & 0 & 1 & 0 & 1 & 0 & 1 & 0 & 0 & 0 & 0 & & & \\
0 & 1 & 0 & 1 & 0 & 1 & 0 & 1 & 0 & 0 & 0 & & & \\
0 & 0 & 1 & 0 & 1 & 0 & 1 & 0 & 1 & 0 & 0 & & & \\
0 & 0 & 0 & 1 & 0 & 1 & 0 & 1 & 0 & 1 & 0 & & & \\
0 & 0 & 0 & 0 & 1 & 0 & 1 & 0 & 1 & 0 & 1 & & & \\
 & & & & & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & & \\
 & & & & & & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \\
 & & & & & & & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot
\end{array}\right),$$

$$(T^3)^{2^2} = T^{12} =$$

$$\left(\begin{array}{cccccccccccccccccccc}
0 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & & & \\
1 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & & & \\
1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & & & \\
1 & 1 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & & & \\
1 & 1 & 1 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & & & \\
1 & 1 & 1 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & & & \\
1 & 1 & 1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & & & \\
0 & 1 & 1 & 1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & & & \\
1 & 0 & 1 & 0 & 1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & & & \\
1 & 1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & & & \\
0 & 1 & 1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & & & \\
0 & 0 & 1 & 1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & & & \\
1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & & & \\
0 & 1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & & & \\
0 & 0 & 1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 1 & 0 & 0 & & & \\
0 & 0 & 0 & 1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 1 & 0 & & & \\
0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 1 & & & \\
 & & & & & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & & \\
 & & & & & & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \\
 & & & & & & & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot
\end{array}\right).$$

To calculate $(T^3)^{2^{t+1}}$ inductively, for each non-negative integer t let $A^t_{i,j}$ be the submatrices partitioning $(T^3)^{2^t}$ such that the size of $A^t_{1,1}$ is

3×3, the size of $A^t_{1,j}$ is 3×2^t where $j > 1$, the size of $A^t_{i,1}$ is $2^t \times 3$ where $i > 1$, the size of $A^t_{i,j}$ is $2^t \times 2^t$ where $i > 1$ and $j > 1$, and $(T^3)^{2^t}$ is partitioned as follows;

$$\begin{pmatrix} A^t_{1,1} & A^t_{1,2} & A^t_{1,3} & A^t_{1,4} & \cdots \\ A^t_{2,1} & A^t_{2,2} & A^t_{2,3} & A^t_{2,4} & \cdots \\ A^t_{3,1} & A^t_{3,2} & A^t_{3,3} & A^t_{3,4} & \cdots \\ A^t_{4,1} & A^t_{4,2} & A^t_{4,3} & A^t_{4,4} & \cdots \\ \vdots & \vdots & \vdots & \vdots & \vdots \end{pmatrix}.$$

Then we can easily have Lemma 3.1.

LEMMA 3.1. *Let T be the characteristic matrix of a uniform CA of sufficiently large length configured with rule* 60 *and intermediate boundary condition. And let $A^t_{i,j}$ be the submatrices partitioning $(T^3)^{2^t}$ as above for each non-negative integer t. Then the submatrices $A^{t+1}_{i,j}$ partitioning $(T^3)^{2^{t+1}}$ as above can be calculated as follows;*

$$A^{t+1}_{i,j} = \begin{cases} \sum_k A^t_{1,k} \cdot A^t_{k,1} & \text{if } i = 1 \text{ and } j = 1, \\ \begin{pmatrix} \sum_k A^t_{1,k} \cdot A^t_{k,2j-2} & \sum_k A^t_{1,k} \cdot A^t_{k,2j-1} \end{pmatrix} & \\ & \text{if } i = 1 \text{ and } j > 1, \\ \begin{pmatrix} \sum_k A^t_{2i-2,k} \cdot A^t_{k,1} \\ \sum_k A^t_{2i-1,k} \cdot A^t_{k,1} \end{pmatrix} & \text{if } i > 1 \text{ and } j = 1, \\ \begin{pmatrix} \sum_k A^t_{2i-2,k} \cdot A^t_{k,2j-2} & \sum_k A^t_{2i-2,k} \cdot A^t_{k,2j-1} \\ \sum_k A^t_{2i-1,k} \cdot A^t_{k,2j-2} & \sum_k A^t_{2i-1,k} \cdot A^t_{k,2j-1} \end{pmatrix} & \\ & \text{if } i > 1 \text{ and } j > 1. \end{cases}$$

If t is an even non-negative integer, then 2^t the column size of $A^t_{i,1}$ with $i > 1$ is $3m+1$ for some non-negative integer m. And if t is an odd non-negative integer, then 2^t the column size of $A^t_{i,1}$ with $i > 1$ is $3m+2$ for some non-negative integer m. Let I and O denote the identity and zero matrices, respectively.

LEMMA 3.2. *Let T be the characteristic matrix of a uniform CA of sufficiently large length configured with rule* 60 *and intermediate boundary condition. And let $A^t_{i,j}$ be the submatrices partitioning $(T^3)^{2^t}$ as above for each non-negative integer t. Then the following* (1), (2), (3), (4), (5) *and* (6) *hold:*

(1) $A^t_{1,1} = \begin{pmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{pmatrix}$.

(2) *All entries of* $A^t_{2,1}$ *are* 1 *so that* $A^t_{2,1} = \begin{pmatrix} 1 & 1 & 1 \\ \vdots & \vdots & \vdots \\ 1 & 1 & 1 \end{pmatrix}$.

(3) *If* t *is even or odd, then* $A^t_{3,1}$ *is an iteration of* $A^t_{1,1} = \begin{pmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{pmatrix}$ *with truncation in the bottom so that*

$$A^t_{3,1} = \begin{pmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \\ \vdots & \vdots & \vdots \\ 0 & 1 & 1 \end{pmatrix} \text{ or } A^t_{3,1} = \begin{pmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \\ \vdots & \vdots & \vdots \\ 1 & 0 & 1 \end{pmatrix},$$

respectively.

(4) *If* t *is even or odd, then* $A^t_{4,1}$ *is an iteration of* $\begin{pmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}$ *or* $\begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix}$ *with truncation in the bottom so that*

$$A^t_{4,1} = \begin{pmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \\ \vdots & \vdots & \vdots \\ 0 & 0 & 1 \end{pmatrix} \text{ or } A^t_{4,1} = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \\ \vdots & \vdots & \vdots \\ 0 & 0 & 1 \end{pmatrix},$$

respectively.

(5) *If* $i > 1$ *and* $j \le i \le j + 3$, *then* $A^t_{i,j} = I$.

(6) *If* $i < j$ *or* $i > j + 3$, *then* $A^t_{i,j} = O$.

Proof. We will use an induction on t. For $t = 0, 1, 2$, the lemma holds by the explicit formulas for $(T^3)^{2^0} = T^3, (T^3)^{2^1} = T^6$ and $(T^3)^{2^2} = T^{12}$ above. Let $t > 2$. For (1),

$$\begin{aligned} A^t_{1,1} &= \sum_k A^{t-1}_{1,k} \cdot A^{t-1}_{k,1} \quad \text{(by Lemma 3.1)} \\ &= A^{t-1}_{1,1} \cdot A^{t-1}_{1,1} \quad \text{(by the induction hypothesis for (6))} \end{aligned}$$

$$= \begin{pmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{pmatrix} \begin{pmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{pmatrix}$$

(by the induction hypothesis for (1))

$$= \begin{pmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{pmatrix}.$$

The upper half part of $A^t_{2,1}$ is

$$\sum_k A^{t-1}_{2,k} \cdot A^{t-1}_{k,1} \quad \text{(by Lemma 3.1)}$$

$$= A^{t-1}_{2,1} \cdot A^{t-1}_{1,1} + I \cdot A^{t-1}_{2,1}$$

(by the induction hypothesis for (5) and (6))

$$= \begin{pmatrix} 1 & 1 & 1 \\ \vdots & \vdots & \vdots \\ 1 & 1 & 1 \end{pmatrix} \begin{pmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{pmatrix} + \begin{pmatrix} 1 & 1 & 1 \\ \vdots & \vdots & \vdots \\ 1 & 1 & 1 \end{pmatrix}$$

(by the induction hypothesis for (1) and (2))

$$= O + \begin{pmatrix} 1 & 1 & 1 \\ \vdots & \vdots & \vdots \\ 1 & 1 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 1 & 1 \\ \vdots & \vdots & \vdots \\ 1 & 1 & 1 \end{pmatrix}.$$

Now we have

$$A^{t-1}_{1,1} + I = \begin{pmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{pmatrix} + \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{pmatrix}$$

by the induction hypothesis for (1). The lower half part of $A^t_{2,1}$ is

$$\sum_k A^{t-1}_{3,k} \cdot A^{t-1}_{k,1} \quad \text{(by Lemma 3.1)}$$

$$= A^{t-1}_{3,1} \cdot A^{t-1}_{1,1} + I \cdot A^{t-1}_{2,1} + I \cdot A^{t-1}_{3,1}$$

(by the induction hypothesis for (5) and (6))

$$= A^{t-1}_{3,1}(A^{t-1}_{1,1} + I) + A^{t-1}_{2,1}$$

$$= \begin{cases} \begin{pmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \\ \vdots & \vdots & \vdots \\ 0 & 1 & 1 \end{pmatrix} \begin{pmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{pmatrix} + \begin{pmatrix} 1 & 1 & 1 \\ \vdots & \vdots & \vdots \\ 1 & 1 & 1 \end{pmatrix} & \text{where } t \text{ is odd,} \\ \begin{pmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \\ \vdots & \vdots & \vdots \\ 1 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{pmatrix} + \begin{pmatrix} 1 & 1 & 1 \\ \vdots & \vdots & \vdots \\ 1 & 1 & 1 \end{pmatrix} & \text{where } t \text{ is even} \end{cases}$$

(by the induction hypothesis for (2) and (3))

$$= O + \begin{pmatrix} 1 & 1 & 1 \\ \vdots & \vdots & \vdots \\ 1 & 1 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 1 & 1 \\ \vdots & \vdots & \vdots \\ 1 & 1 & 1 \end{pmatrix}.$$

Thus (2) holds. The upper half part of $A^t_{3,1}$ is

$$\sum_k A^{t-1}_{4,k} \cdot A^{t-1}_{k,1} \quad \text{(by Lemma 3.1)}$$

$$= A^{t-1}_{4,1} \cdot A^{t-1}_{1,1} + I \cdot A^{t-1}_{2,1} + I \cdot A^{t-1}_{3,1} + I \cdot A^{t-1}_{4,1}$$

(by the induction hypothesis for (5) and (6))

$$= A^{t-1}_{4,1}(A^{t-1}_{1,1} + I) + A^{t-1}_{2,1} + A^{t-1}_{3,1}$$

$$= \begin{cases} \begin{pmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \\ \vdots & \vdots & \vdots \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{pmatrix} + A^{t-1}_{2,1} + A^{t-1}_{3,1} & \text{where } t \text{ is odd,} \\ \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \\ \vdots & \vdots & \vdots \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{pmatrix} + A^{t-1}_{2,1} + A^{t-1}_{3,1} & \text{where } t \text{ is even} \end{cases}$$

(by the induction hypothesis for (4))

$$= \begin{pmatrix} 1 & 1 & 1 \\ \vdots & \vdots & \vdots \\ 1 & 1 & 1 \end{pmatrix} + A_{2,1}^{t-1} + A_{3,1}^{t-1}$$

$$= \begin{pmatrix} 1 & 1 & 1 \\ \vdots & \vdots & \vdots \\ 1 & 1 & 1 \end{pmatrix} + \begin{pmatrix} 1 & 1 & 1 \\ \vdots & \vdots & \vdots \\ 1 & 1 & 1 \end{pmatrix} + A_{3,1}^{t-1} = A_{3,1}^{t-1}$$

(by the induction hypothesis for (2)).

Thus, by the induction hypothesis for (3), the upper half part of $A_{3,1}^t$ is

$$\begin{pmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \\ \vdots & \vdots & \vdots \\ 1 & 0 & 1 \end{pmatrix} \quad \text{or} \quad \begin{pmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \\ \vdots & \vdots & \vdots \\ 0 & 1 & 1 \end{pmatrix}$$

where t is even or odd, respectively. The lower half part of $A_{3,1}^t$ is

$$\sum_k A_{5,k}^{t-1} \cdot A_{k,1}^{t-1} \text{ (by Lemma 3.1)}$$

$$= I \cdot A_{2,1}^{t-1} + I \cdot A_{3,1}^{t-1} + I \cdot A_{4,1}^{t-1}$$

(by the induction hypothesis for (5) and (6))

$$= \begin{cases} \begin{pmatrix} 1 & 1 & 1 \\ \vdots & \vdots & \vdots \\ 1 & 1 & 1 \end{pmatrix} + \begin{pmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \\ \vdots & \vdots & \vdots \\ 1 & 0 & 1 \end{pmatrix} + \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \\ \vdots & \vdots & \vdots \\ 0 & 0 & 1 \end{pmatrix} & \text{where } t \text{ is even,} \\ \begin{pmatrix} 1 & 1 & 1 \\ \vdots & \vdots & \vdots \\ 1 & 1 & 1 \end{pmatrix} + \begin{pmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \\ \vdots & \vdots & \vdots \\ 0 & 1 & 1 \end{pmatrix} + \begin{pmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \\ \vdots & \vdots & \vdots \\ 0 & 0 & 1 \end{pmatrix} & \text{where } t \text{ is odd} \end{cases}$$

(by the induction hypothesis for (2), (3) and (4)).

Thus the lower half part of $A_{3,1}^t$ is

$$\begin{pmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \\ 1 & 0 & 1 \\ \vdots & \vdots & \vdots \\ 0 & 1 & 1 \end{pmatrix} \quad \text{or} \quad \begin{pmatrix} 1 & 0 & 1 \\ 1 & 1 & 0 \\ 0 & 1 & 1 \\ \vdots & \vdots & \vdots \\ 1 & 0 & 1 \end{pmatrix}$$

where t is even or odd, respectively. So (3) holds. The upper half part of $A^t_{4,1}$ is

$$\begin{aligned}
&\sum_k A^{t-1}_{6,k} \cdot A^{t-1}_{k,1} \qquad \text{(by Lemma 3.1)} \\
&= I \cdot A^{t-1}_{3,1} + I \cdot A^{t-1}_{4,1} \text{ (by the induction hypothesis for (5) and (6))} \\
&= \begin{cases} \begin{pmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \\ \vdots & \vdots & \vdots \\ 1 & 0 & 1 \end{pmatrix} + \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \\ \vdots & \vdots & \vdots \\ 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \\ \vdots & \vdots & \vdots \\ 1 & 0 & 0 \end{pmatrix} & \text{where } t \text{ is even,} \\ \begin{pmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \\ \vdots & \vdots & \vdots \\ 0 & 1 & 1 \end{pmatrix} + \begin{pmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \\ \vdots & \vdots & \vdots \\ 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \\ \vdots & \vdots & \vdots \\ 0 & 1 & 0 \end{pmatrix} & \text{where } t \text{ is odd} \end{cases} \\
&\qquad\qquad\qquad \text{(by the induction hypothesis for (3) and (4)).}
\end{aligned}$$

And the lower half part of $A^t_{4,1}$ is $\sum_k A^{t-1}_{7,k} \cdot A^{t-1}_{k,1} = I \cdot A^{t-1}_{4,1} = A^{t-1}_{4,1}$ by Lemma 3.1 and the induction hypothesis for (5) and (6). So (4) holds by the induction hypothesis for (4). And (5) and (6) hold obviously. $\square$

Let T_n be the characteristic matrix of a uniform CA of length $n \geq 3$ configured with rule 60 and intermediate boundary condition. For a positive integer s, let M_s be the matrix of size $s \times 3$ with all entries are 1. For an integer $n \geq 4$, let P_n be the matrix of size $n \times n$ partitioned by 4 matrices T_3, O, M_{n-3} and I so that $P_n = \begin{pmatrix} T_3 & O \\ M_{n-3} & I \end{pmatrix}$. And let $P_3 = T_3$. Then we can have $(P_n)^2 = P_n$ and $T_n P_n = T_n$ for all integers $n \geq 3$ with direct computation. And if $3 \leq n \leq 3 + 2^t$ for some non-negative integer t then $(T_n)^{2^t \cdot 3} = P_n$ by Lemma 3.2. Thus we have Theorem 3.3.

THEOREM 3.3. *Let T be the characteristic matrix of a uniform CA of length n with $3 \le n \le 3 + 2^t$ for some non-negative integer t configured with rule 60 and intermediate boundary condition. Then $(T^{2^t \cdot 3})^2 = T^{2^t \cdot 3}$ and $T^{1+2^t \cdot 3} = T$.*

Combining this theorem with Theorem 2.3, we have Theorem 3.4.

THEOREM 3.4. *Let T be the characteristic matrix of a uniform CA of length n with $3 + 2^{t-1} < n \le 3 + 2^t$ for some positive integer t configured with rule 60 and intermediate boundary condition. Then $T^{1+m} = T$ for some positive integer m if and only if m is a multiple of $2^t \cdot 3$.*

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Department of Mathematics
Kyungsung University
Busan 608-736, Republic of Korea
E-mail: jgkim@ks.ac.kr